Neural Network-Based Software for Fertilizer Optimization in Precision Farming

Dragoljub Pokrajac, Zoran Obradovic

Center for Information Science and Technology
Temple University
303 Wachman Hall (038-24), 1805 N. Broad St., Philadelphia, PA 19122, USA
{pokie,zoran}@ist.temple.edu

Abstract

A novel technique for providing fertilizer recommendation in precision agriculture is proposed. The method is based on the maximization of the profit function approximated using a decision support system based on artificial neural networks. The software implementation of the proposed approach is described and its use is illustrated on simulated realistic data. Experimental results suggest that the proposed technique is applicable for site-specific crop management.

1 Introduction

The production of maximal crop quantity, with an optimal cost and with restricted use of potentially hazardous materials such as fertilizers and pesticides, is the primary aim in agriculture [6]. According to the economic law of diminishing returns [9], there is an optimum quantity of fertilizer that maximizes profit in agricultural business. On the other side, in the past several years, the issue of environmental protection received a lot of attention in agriculture [13] and emphasized the marginal social cost of pollution [9] as an additional important factor in agricultural management.

Classic agriculture management considers the application of the constant fertilization rate to the whole cultivated field [11]. This uniform fertilization rate is usually determined based on a limited soil sampling and an expert knowledge from fertilization recommendations. However, it has been demonstrated that by applying a variable fertilization rate, profit can be increased due to improved efficiency of resource management. For instance, Long et al. [16] have shown that by varying the concentration of applied nitrogen using a simple linear model, input efficiency of the fertilized nutrient can be improved resulting in increasing profitability. With challenging new technical advances, such as Global Positioning System (GPS), it is possible to gather a lot of data and perform more complex non-linear modeling of crop yield dependence on nutrients. Recently, Soltanpour et al. [25] employed non-linear regression models to calculate nitrogen fertilizer recommendation rates for different yield goals and various cost/price ratios. However, they did not attempt to optimize fertilization of multiple nutrients and to apply their results in a site-specific setting although Watkins et al. [28] have argued that variable rate application of one input alone might be unprofitable and anticipated the necessity for this multivariate fertilization optimization.

The objective of this study is to provide a tool for optimizing financial gain in agricultural business through a site-specific fertilizer application of multiple nutrients, by using neural network-based non-linear optimization techniques [12]. In the following sections we introduce methodology, describe software implementation and illustrate the effects of performed site-specific fertilization recommendations on simulated data sets.

2 Methodology

2.1 Problem Statement

A crop yield is considered to be a non-linear function of controllable features (e.g. concentrations of various nutrients, and irrigation intensity) as well as of non-controllable features (e.g. terrain attributes such as slope and profile curvature) on a two-dimensional field F. At each sampling location $\mathbf{s}=(x,y)\in F$ the concentration of *controllable* features $f_1(\mathbf{s}),\ldots, f_m(\mathbf{s})$ can be increased in a fertilization process by application of treatments $\Delta f_i(\mathbf{s})\geq 0$ $i=1,\ldots,m$, whereas the values of n *uncontrollable* features $f_{m+1}(\mathbf{s}),\ldots, f_{m+n}(\mathbf{s})$ are independent of the treatment procedure. The goal is to determine non-negative treatments $\Delta f_1,\ldots,\Delta f_m$ that maximize total profit on the field, defined as:

$$Profit(\Delta f) = \iint_F (c\Delta Y(\mathbf{s}) - \sum_{i=1,m} w_i \Delta f_i(\mathbf{s}) - w_0)\, d\mathbf{s} \quad (1)$$

Here, c is the unit price of crop, $\Delta Y(\mathbf{s})$ is increment of crop yield due to treatments, $\mathbf{w}=[w_1\ w_2 \ldots w_m]^T$ is a vector of prices per unit of a particular feature and w_0 is a fixed unit cost of agriculture management.

In addition, we define the average profit per a unit area due to particular treatments $\Delta\mathbf{f}$ as

$$\overline{Profit(\Delta\mathbf{f})} \equiv Profit(\Delta\mathbf{f}) / area(F) \quad (2)$$

and the average cost of fertilization

$$\overline{Cost(\Delta\mathbf{f})} = \frac{1}{area(F)} \iint_F \Big(\sum_{i=1,m} w_i \, \Delta f_i(\mathbf{s}) + w_0 \Big) \, ds \quad (3)$$

Due to additivity of (1), the profit on field F is optimized if a localized profit $p(\mathbf{s}) = c\Delta Y(\mathbf{s}) - \sum_{i=1,m} w_i \, \Delta f_i(\mathbf{s}) - w_0$ is maximized on every point s of the field, resulting in a vector of optimal treatments

$$\Delta\mathbf{f}_{opt}(\mathbf{s}) = [\Delta f_{1,opt}(\mathbf{s}) \; \Delta f_{2,opt}(\mathbf{s}) \ldots \Delta f_{m,opt}(\mathbf{s})]^T \quad (4)$$

that satisfy

$$\Delta\mathbf{f}_{opt}(\mathbf{s}) = \underset{\substack{\Delta f_1(\mathbf{s}), \Delta f_2(\mathbf{s}), \ldots, \Delta f_m(\mathbf{s}), \\ \Delta f_i(\mathbf{s}) \geq 0, i=1,\ldots,m}}{\arg\max} p(\mathbf{s}). \quad (5)$$

The result of a profit optimization procedure is the estimated treatment vector:

$$\Delta\hat{\mathbf{f}}(\mathbf{s}) = [\Delta\hat{f}_1(\mathbf{s}) \quad \Delta\hat{f}_2(\mathbf{s}) \quad \ldots \quad \Delta\hat{f}_m(\mathbf{s})]^T \quad (6)$$

which is ideally equal to $\Delta\mathbf{f}_{opt}(\mathbf{s})$.

In practice, crop yield and features are available only on a finite set of sampling points s within the field. In addition, due to finite accuracy of devices for on-the-field fertilizer application [4] feasible treatment rates can be treated as discrete:

$$\Delta\hat{f}_i(\mathbf{s}) \in \{0, \delta f_i, 2\delta f_i, \ldots, \Delta f_{i,\max}\}, i = 1,\ldots,m. \quad (7)$$

Here, δf_i is the minimal portion by which the fertilization rate of the i-th controllable feature can be adjusted, dependent on technological characteristics of the fertilizer applicator, and $\Delta f_{i,\max}$ is the maximal allowed rate, related to environmental considerations [1], [23].

2.2 Optimization Method

The problem of fertilizer recommendation is a special case of the constrained maximization [27]. However, here the localized profit function p is not known in the closed form, so the problem cannot be solved using the theory of non-linear constrained optimizations [18]. To optimize this incompletely specified functional dependence in this paper we propose direct and inverse modeling, for optimization of a profit function approximation.

In *direct modeling*, the crop yield Y at each sampling point is treated as a function of both controllable and non-controllable features. The function

$$Y(\mathbf{s}) = Y(f_1,\ldots,f_m, f_{m+1},\ldots,f_{m+n}) \quad (8)$$

is estimated using a non-parametric regression model, and the estimate $\hat{Y}(\mathbf{s})$ is subsequently employed for constrained maximization of the following determinant function:

$$d_i(f_1', f_2', \ldots, f_{m+n}') = \hat{Y}(\mathbf{s}) - \sum_{i=1,m} w_i^* f_i'(\mathbf{s}) \quad (9)$$

Here,

$$\begin{aligned} f_i' &\geq f_i \;, i = 1,\ldots,m \\ f_i' &= f_i \;, i = m+1,\ldots,m+n \end{aligned} \quad (10)$$

w_i^*, i=1,…,m are cost/price ratios for each controllable feature-fertilized nutrient [25], and the estimated treatments, eq. (6), are obtained as differences

$$\Delta\hat{f}_i(\mathbf{s}) = f_i' - f_i, i = 1,\ldots,m \quad (11)$$

where $f_i', i = 1,\ldots,m$ are the results of constrained maximization of the determinant (9).

Using the direct modeling, a profit as a function of controllable features can be maximized by independent as well as by simultaneous optimization of site-specific treatment rates. In *independent optimization*, the optimal fertilizer concentration is obtained for each nutrient separately. For each controllable feature f_i, i=1,…,m, the estimated treatment rate is obtained using eq. (11), where the value of *i-th* feature f_i' is chosen to maximize the discriminant value $d_i(f_1, \ldots f_{i-1}, f_i', f_{i+1}, \ldots, f_{m+n})$ under the constraint $f_i' \geq f_i$, while the other features retaining their initial values. Independent optimization can in principle be performed using a linear search [22]. However, due to (7), it is sufficient to perform exhaustive search on a finite and fairly small set of allowed fertilization rates. Independent optimization works best when crop yield is a separable function of features that do not interact significantly.

Simultaneous optimization aims towards a global maximization of financial gain as a function of all administered nutrients. Using standardized optimization techniques (e.g. [5]), through an iterative process, all the values f_i', i=1,…,m are simultaneously updated towards an increase of the determinant function. The main drawback of this method is potential danger of detecting a local instead of the global maximum and sensitivity on initial values for f_i'. Also, due to nature of optimization algorithms, this technique may be more time consumptive compared to the independent optimization.

In *inverse modeling,* controllable features are regressed on crop yield and on non-controllable features. Since the same value of crop yield may correspond to different tuples of feature values, here we can model only the dependence of one selected controllable feature f_i ($1 \le i \le m$) on the other features and yield. Hence, in inverse modeling, at each spatial location **s** we estimate the function

$$f_i(\mathbf{s}) \equiv g(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}, Y). \quad (12)$$

When an estimate $\hat{g}(Y)$ of function (12) is computed, the desirable level $f_i^{'} = \hat{g}(Y)$ of the i-th controllable feature is obtained to correspond to the crop yield that maximize the following discriminant function

$$d_i(Y) = Y - w_i^* \cdot \hat{g}(Y). \quad (13)$$

When fertilization rates are being optimized for multiple features, the estimated rates are obtained through an independent optimization of each controllable feature, using equations (12) and (13).

Observe that both direct and inverse modeling introduced here involve an estimation of regression models, which can be done using various parametric and non-parametric techniques. In this paper, in addition to linear regression using ordinary least squares [8], we advocate an application of multi-layer feed-forward neural networks with sigmoidal and radial-basis (RBF) activation functions [12]. The topology of applied multi-layer neural networks is shown in Fig. 1. The networks have $m+n$ inputs, and the output is a linear function of K hidden neuron activation functions $h_k(x_1, \ldots, x_{m+n})$, $k=1,\ldots,K$. In multi-layer perceptrons, (MLP), activation functions h_k are logistic sigmoids:

$$h_k(x_1, \ldots, x_{m+n}) = 1/1 + \exp\left(-v_{k,0} + \sum_{i=1}^{m+n} v_{k,i} x_i\right) \quad (14)$$

In radial basis functions network (RBF), the k-th activation function depends on the Mahalonobis distance between the input vector $\mathbf{x}=[x_1,\ldots,x_{m+n}]^T$ and a vector $\mathbf{c}_k=[c_{1,k},\ldots,c_{m+n,k}]^T$ that determines the center of the k-th basis function. The shape of the function is specified by a positive-definite matrix $\mathbf{\Sigma}_k$. More precisely,

$$h_k(x_1, \ldots, x_{m+n}) = \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{c}_k)^T \mathbf{\Sigma}_k^{-1} (\mathbf{x} - \mathbf{c}_k)\right) \quad (15)$$

Applied multi-layer neural networks can be trained using standard learning algorithms ([12], [17], [19]).

2.3 Performance Evaluation

When the fertilization recommendation is performed on real-world data, the true effects of the treatment can be

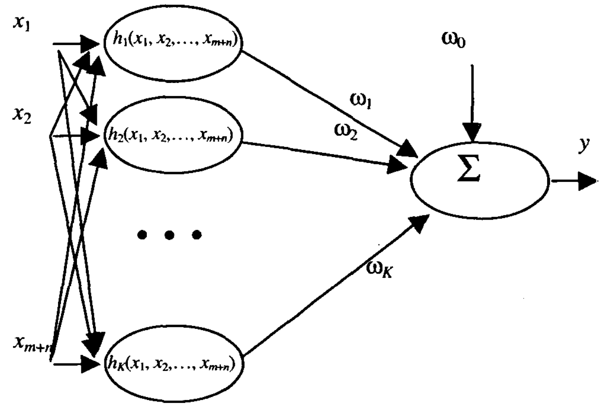

Figure 1: Topology of applied neural networks

determined using single-factor analysis of variance [8]. However, these solutions are expensive and require the implementation of randomization strategies for experimental fields, which is difficult to properly accomplish when the fields are heterogeneous (e.g. with high variability of soil types and a complex topography). In contrast, for simulated data, we can compare obtained recommendations with known optima and thus determine true quality of provided estimates.

There are several ways to evaluate the quality of fertilizer recommendations. Thus, one can estimate the coefficient of determination R^2 [8] as a measure of similarity between optimal and estimated treatments $\Delta\hat{f}_{i,opt}(\mathbf{s})$ and $\Delta\hat{f}_i(\mathbf{s})$ for the i-th controllable feature on the field. However, R^2 is not suitable when fertilizer recommendation models are biased (i.e. when averaged optimal and estimated treatments differ) since in this case, coefficient of determination can be negative, although the proposed fertilization recommendations may be still acceptable.

Another possibility is to use Pearson's coefficient of correlation r between the predicted and optimal fertilization [8]. The coefficient of correlation measures the strength of the linear relationship between predicted and optimal fertilization rate but has a limited use when this dependence is non-linear.

Observe that both r and R^2 portray the quality of recommendations for each nutrient independently instead of providing a global assessment of the adopted fertilization policy. To rectify this problem and enable the comparison of the site-specific treatment recommendations with a uniform fertilization when the same total amounts of fertilizers are spent, we propose a comparison of the average profit $\overline{Profit(\Delta\hat{\mathbf{f}})}$, computed using eq. (2) when the recommended treatments $\Delta\hat{f}_i, i = 1,\ldots,m$ are applied, with the mean profit $\overline{Profit(\Delta\hat{\mathbf{f}}_{avg})}$ computed when the averaged recommended fertilization rates

$$\Delta f_{i,avg} = \frac{1}{area(F)} \iint_F \Delta f_i(\mathbf{s}) ds \quad ,i = 1,\ldots,m \qquad (16)$$

are applied uniformly on the whole field. In addition, we compare these averaged profits with the optimal average profit $\overline{Profit(\Delta\mathbf{f}_{opt})}$ achievable through the optimal fertilization rates, eq. (5).

3 Implementation

The proposed method is implemented in MATLAB [10] to accomplish a user-friendly process of treatment recommendations. The software consists of three modules: Model Training, Treatment Prediction and Visualization.

In *Model Training*, user selects the type of modeling (direct or inverse) and performs the training of the chosen regression model. The user can specify names of input files containing data for parameter estimation, relevant features that influence the crop yield and the technique of regression model estimation. In the current version of the software, we support multi-layer networks with logistic (MLP) and radial basis (RBF) transfer functions of hidden neurons and linear ordinary least-squares (OLS) models. For a regression model, the user can specify model parameters and choose the learning algorithm. For instance, when using RBF, in addition to a simple learning algorithm with a fixed number and parameters of radial basis functions, we support self-organized maps [15] and regression trees [19] for an automatic initialization of hidden layer parameters (see Fig. 2). When MLP networks are used, the input parameters include the number of hidden neurons and the maximal number of epochs as well as a choice among various learning algorithms such as Polak-Ribiere conjugate-gradient, Quasi-Newton, resilient backpropagation and Levenberg-Marquardt algorithm [7].

The *treatment prediction* module is activated automatically once regression model coefficients are estimated in *Model Training*. In addition, the user can started this module with loaded pre-computed coefficients. In this mode, the user specifies controllable features and an optimization algorithm (independent or simultaneous optimization). Before the computation of treatment prediction actually begins, the user should also specify treatment parameters for each feature (nutrient): maximum fertilizer rate $\Delta f_{i,\max}$, fertilizer unit cost w_i, and application resolution $\delta f_i, i = 1,\ldots,m$. Additional input parameters are the constant cost of fertilization w_0 (independent of the quantity of applied fertilizer) and the unit price of crop c

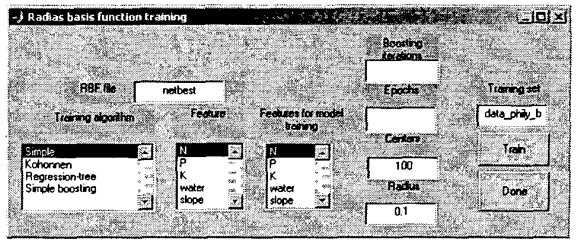

Figure 2: Parameters of a RBF regression model

(see Fig. 3). Depending on the size of the field, the number of controllable features, the application resolution and the type of algorithm (independent or simultaneous optimization), the computation of estimated fertilization rates can elapse from 10 minutes to 2-3 hours on a Pentium III processor with 866MHz and 256MB RAM. The user can keep track of the estimation progress and stop or pause the computing. The final output in this mode is the file with estimated treatments rate for each controllable features on every discretized spatial location of the field.

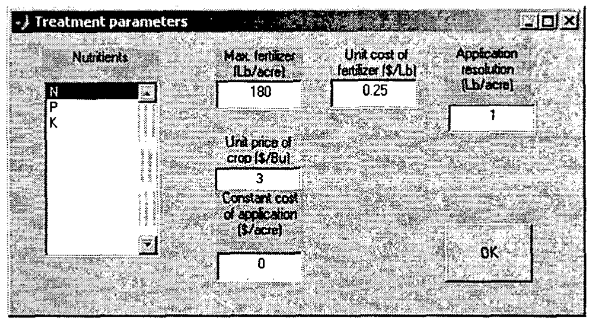

Figure 3: Parameters for treatment recommendation

In *Visualization* mode, the user can inspect the properties of the trained regression model and examine the estimated fertilizer recommendations. When the experiments are performed on simulated data, the user can also compare the obtained results with a known optimum and thus evaluate the performed estimation process. For a chosen controllable feature, the user can visualize final results as the maps of estimated and optimal fertilization rates and compare average profits on the field achievable using recommended, uniform and the optimal fertilization.

4 Experimental Results

The proposed techniques provide potentials to explore various issues of crop management in agriculture. Due to space limitations, in this paper we only demonstrate the applicability of the direct modeling technique on simulated agricultural data. More extensive experiments will appear in a forthcoming paper [21].

For experimental evaluation, data were generated using our simulator of spatial data [20] and consisted of two fields – the training and the test dataset. Each dataset consisted of simulated crop yield and five features representing soil nitrogen, phosphorus, and potassium, land slope and profile curvature. The spatial statistics of the features roughly corresponded to ranges obtained from a real-world data set [14] and all features were approximately normally distributed. Data were generated on a 10m sampling grid and the size of each field was 800m $\times$800m, such that each simulated field consisted of 6561 examples. Yield was generated using a plateau model [20] with parameters set using fertilizer recommendation guides [3] and regression results from a real-world data set [14].

Using the direct modeling, we examined the applicability of neural networks for fertilizer recommendation and compared networks with radial basis (RBF) and sigmoidal (MLP) hidden neurons to ordinary least squares (OLS) linear models. In RBF networks, the hidden layer consisted of 100 hidden neurons. Matrices $\mathbf{\Sigma}_k$ were pre-specified and identical for all hidden neurons while centers $\mathbf{c}_k$ were randomly selected from the training set. MLP networks had 10 hidden neurons and were trained using the Levenberg-Marquardt algorithm. Since the purpose of experiments was not to determine the ultimate performance of the proposed methods, no further optimization of network structure and parameters has been attempted.

After the training of regression models, the fertilization rates for nitrogen, phosphorus, and potassium were estimated on the test field using the techniques of independent and simultaneous optimization. We employed the following values of treatment parameters. Maximum fertilizer rates were 320, 48, and 272 lb/acre (181, 27, and 154 kg/ha) per nitrogen, phosphorus and potassium, respectively, fertilizer unit cost was $0.25/lb (55c/kg), and the resolution of fertilizer application was 1 lb/acre (0.565 kg/ha) for all nutrients. We assumed the unit crop price of $3/Bu of wheat (11cents/kg) and a zero constant cost of fertilization. Due to stochastic character of the applied algorithm for learning neural network coefficients, corresponding experiments were repeated 10 times.

Experimental results suggest the applicability of the proposed direct modeling method for fertilization rate recommendations. Since the plateau model employed to simulate crop yield does not introduce feature interactions, independent and simultaneous optimization techniques resulted with similar effects. The obtained fertilization rates using neural networks were correlated with the optimal rates (coefficient of correlation larger than 0.5) but due to the bias of applied non-linear models, the average recommended rates differ from the average optimal ones, which sporadically resulted with a negative coefficient of determination.

At Table 1, comparison of the averaged fertilization costs and profit obtained using uniform and site-specific fertilization rates (estimated by direct modeling and independent optimization technique) to the known optimum is presented. Site-specific fertilization rates provided using neural networks resulted with an average profit close to the optimal value. For instance, using RBF networks, we achieved average profit of $210/acre compared to the average optimal profit of $247/acre. The profit was similar using either MLP or RBF networks, but the networks with radial basis neurons generally provided treatment recommendations with a smaller fertilization cost and hence reduced environmental impacts. This was probably caused by the known tendency of radial basis functions to not extrapolate far from training examples [12].

When neural network regression models were applied, the application of site-specific treatment rates instead of a uniform fertilization resulted with significant profit gains. Thus, the same total amounts of fertilizer resulted with an average profit of $169/acre when applied uniformly and with $213/acre when applied according to the treatment recommendations obtained with MLP networks. In contrast, the application of the proposed optimization methods with linear OLS models did not result with significant improvements compared to uniform fertilization. Furthermore, the application of linear models was inferior to neural networks. For instance, by using OLS average profit per acre was $42 smaller, while the average cost of fertilization was $74 more than in a site-specific fertilization by RBF networks.

5 Conclusions and Work in Progress

In this paper we presented a method for neural-network based optimization of fertilization rates in precision agriculture and described a software for providing site-specific treatment recommendations based on direct and inverse modeling of crop yield.

Preliminary experimental results suggest that the proposed technique has a high potential to significantly increase financial gain and substantially reduce average fertilization rates compared to the traditionally performed uniform fertilization and the application of linear regression models.

Work in progress includes theoretical and practical modeling aspects, such as the choice of modeling type (direct/inverse, independent/simultaneous optimization) and comparison of particular regression models

In addition, using the proposed techniques, we currently explore important aspects of fertilization rate estimation that arise in the agriculture practice such as the influence of unobserved features relevant for yield prediction, the significance of sampling resolution and presence of sensor and measurement errors in features and yield ([2], [24],

[26]) as well as the impact of cost/price ratio on the optimized fertilization rates.

Table 1: Comparison of averaged fertilization cost and profit of the uniform and site-specific fertilization with rates estimated by direct modeling and independent optimization. Results obtained by ordinary-least squares (OLS) and neural networks with sigmoidal (MLP) and radial-basis (RBF) hidden neurons are compared with a known optimum on simulated data.

Treatment			Average fertilization cost ($/acre)	Average profit ($/acre)
Estimated	OLS	Uniform	128	167
		Site-specific		168
	MLP	Uniform	82	169
		Site-specific		213
	RBF	Uniform	54	185
		Site-specific		210
Optimal			50	247

6 References

[1] Bacon, P.E. (ed.), *Nitrogen Fertilization in the Environment*, Marcel Dekker Inc., New York, 1995.

[2] S. Blackmore and C. Marshall, "Yield Mapping, Errors and Algorithms", *Proc. 3rd Int'l Conf Precision Agriculture*, ASA-CSSA-SSSA, Madison, WI, 1996, pp. 403-415.

[3] Brown, B., *Idaho Fertilizer Guide-Irrigated Wheat*, Current Information Series (CIS) No 373, University of Idaho, College of agriculture, Cooperative extension service, Agriculture experimental station, Moscow, ID, 1982.

[4] R.L Clarke, and R.L. McGuckin, "Variable Rate Application Technology: An Overview", *Proc. 3rd Int'l Conf Precision Agriculture*, ASA-CSSA-SSSA, Madison, WI, 1996, pp. 855-862.

[5] T.F Coleman, and Y. Li, "An Interior, Trust Region Approach for Nonlinear Minimization Subject to Bounds", *SIAM Journal on Optimization*, Vol. 6,1996, pp. 418-445.

[6] Cramer, G.L., C.W. Jensen, and D. D. Southgate, *Agricultural Economics and Agribusiness*, 8th edn, John Wiley & Sons, New York, NY, 2000.

[7] Demuth, H., and M. Beale, *Neural Network Toolbox for use with MATLAB, Users Guide,* Version 3, The MathWorks, Inc., Natick, MA, 1998.

[8] Devore, J.L., *Probability and Statistics for Engineering and the Sciences*, 4th edn, International Thomson Publishing Company, Belmont, CA, 1995.

[9] Dolan, E.G., *Basic Economics,* 2nd edn, The Dryden Press, Hinsdale, IL, 1980.

[10] Hanselman, D. C., and B. C. Littlefield, *Mastering MATLAB 5: A Comprehensive Tutorial and Reference*, Prentice Hall, Upper Saddle River, NJ, 1997.

[11] Havlin, J. (ed.), S. L. Tisdale (ed.), and J. D. Beaton, *Soil Fertility and Fertilizers: An Introduction to Nutrient Management*, 6th edn, Prentice Hall, Upper Saddle River, NJ, 1998.

[12] Haykin, S., *Neural Networks: A Comprehensive Foundation,* 2nd edn, Prentice Hall, Upper Saddle River, NJ, 1999.

[13] G. Hergert, W. Pan, D. Huggins, J. Grove, and T. Peck, "Adequacy of Current Fertilizer Recommendation for Site-Specific Management", In *The State of Site-Specific Management for Agriculture*, ASA/CSSA/SSSA, Madison WI, 1997, pp. 283-300.

[14] J.R. Hess, and R.L. Hoskinson, "Methods for Characterization and Analysis of Spatial and Temporal Variability for Researching and Managing Integrated Farming Systems", *Proc. 3rd Int'l Conf Precision Agriculture*, ASA-CSSA-SSSA, Madison, WI, 1996, pp. 641-650.

[15] Kohonen, T., *Self-organization and Associative Memory*, Springer-Verlag, New York, NY, 1984.

[16] D. Long, G.R. Carlson, G.A. Nielsen, and G. Lachapelle, "Increasing Profitability With Variable Rate Fertilization", *Montana AgResearch*, MSU Communications Services, Bozeman, MT, 1995. (www.montana.edu/wwwpb/ag/long.html).

[17] D. Lowe, "Adaptive Radial Basis Function Nonlinearities, and the Problem of Generalization", *Proc. 1st IEE Int'l Conf. on Artificial Neural Networks*, 1989, pp. 171-175.

[18] Nocedal, J., and S. J. Wright, *Numerical Optimization*. Springer, New York, NY, 1999.

[19] M. Orr, J. Hallam, K. Takezawa, A. Murray, S. Ninomiya, M. Oide and T. Leonard, "Combining Regression Trees and Radial Basis Function Networks", *International Journal of Neural Systems*, Vol. 10, No. 6, 2000, pp. 453-466,

[20] D. Pokrajac, T. Fiez, and Z. Obradovic, "A Tool for Controlled Knowledge Discovery in Spatial Domains", *Proc. 14th European Simulation Multiconference (ESM)*, 2000.

[21] D. Pokrajac, and Z. Obradovic, "A Neural Network-Based Method for Site-Specific Fertilization Recommendation", *Proc. ASAE annual meeting* 2001, in press.

[22] Press, W., S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edn, Cambridge Univ. Press, Cambridge, UK, 1993.

[23] Rechcigl, J.E, "Phosphorus-Natural versus Pollution Levels: Lake Okeechobee Case Study", *46th Proc. Annual Florida Beef Cattle Short Course*, Gainesville, FL, 1997, p.61. (www.animal.ufl.edu/short97/Rechcig2.pdf).

[24] S.P. Saunders, G. Larscheid, B.S. Blackmore, and J.V. Stafford, "A method for Direct Comparison of Differential Global Positioning System Suitable for Precision Farming", *Proc. 3rd Int'l Conf Precision Agriculture*, ASA-CSSA-SSSA, Madison, WI, 1996, pp. 663-680.

[25] P.N, Soltanpour, A. Schlegel, G. Cardon, R. Waskom, and A. Halvorson, "Regional N Fertilizer Recommendations for Dryland Wheat", In *Proc. 2000 Great Plains Soil Fertility Conference*, Vol. 8, 2000, pp. 335-340.

[26] J.V. Stafford, B. Ambler, R.M. Lark, and J. Catt, "Mapping and Interpreting the Yield Variation in Cereal Crops", *Computers and Electronics in Agriculture*, Vol. 14, No. 2/3, 1996, pp. 101-119.

[27] Sundaram, R.K., *A First Course in Optimization Theory*, Cambridge University Press, New York, NY, 1996.

[28] K. Watkins, Y. Lu, and W. Huang, "Economic and Environmental Feasibility of Variable Rate Nitrogen Fertilizer Application with Carry-Over Effects", *J. Agricultural and Resource Economics*, Vol. 23, No. 2, 1998, pp. 401-426.